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Isomorphism Theorems for QA-Mappings

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Abstract

In this paper we introduce the concept of QA-mappings and Q-quasi-antiorders in anti-ordered sets theory. Two isomorphism theorems for QA-mappings and Q-quasi-antiorders are presented.

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1 Introduction

Let $(X, =, \neq)$ be a set in the sense of books [1] - [3] and [10], where " $\neq$ " is a binary relation on X which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq z \Longrightarrow x \neq y \vee y \neq z,$$
$$x \neq y \wedge y = z \Longrightarrow x \neq z,$$

called *apartness* (A. Heyting). The apartness is *tight* (D.Scott) if $\neg(x \neq y) \Longrightarrow x = y$ holds. Let Y be a subset of X and $x \in X$. The subset Y of X is *strongly extensional* in X if and only if $y \in Y \Longrightarrow y \neq x \vee x \in Y$ ([3],[5]). If $x \in X$, it defined ([2]) $x \bowtie Y$ by $(\forall y \in Y)(y \neq x)$.

Let $f : (X, =, \neq) \longrightarrow (Y, =, \neq)$ be a function. We say that it is:
(a) f is *strongly extensional* if and only if $(\forall a, b \in X)(f(a) \neq f(b) \Longrightarrow a \neq b)$;
(b) f is an *embedding* if and only if $(\forall a, b \in X)(a \neq b \Longrightarrow f(a) \neq f(b))$.
Let $\alpha \subseteq X \times Y$ and $\beta \subseteq Y \times Z$ be relations. The *filled product* ([4]) of relations α and β is the relation

$$\beta * \alpha = \{(a, c) \in X \times Z : (\forall b \in Y)((a, b) \in \alpha \vee (b, c) \in \beta)\}.$$

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A relation $q \subseteq X \times X$ is a *coequality* relation on X if and only if holds:

$$q \subseteq \neq,\ q \subseteq q^{-1},\ q \subseteq\ q * q.$$

If q is a *coequality relation* on set $(X, =, \neq)$, we can construct factor-set $(X/q, =_1, \neq_1)$ with

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q,\ aq \neq_1 bq \Longleftrightarrow (a, b) \in q.$$

A relation α on X is *antiorder* ([6],[7]) on X if and only if

$$\alpha\ \subseteq \neq,\ \alpha\ \subseteq\ \alpha * \alpha,\ \neq \subseteq \alpha \cup \alpha^{-1}, (\alpha \cap \alpha^{-1} = \emptyset).$$

Let $f : (X, =, \neq, \alpha) \longrightarrow (Y, =, \neq, \beta)$ be a strongly extensional function of ordered sets under antiorders. f is called *isotone* if

$$(\forall x, y \in S)((x, y) \in \alpha \Longrightarrow (f(x), f(y)) \in \beta);$$

f is called *reverse isotone* if and only if

$$(\forall x, y \in S)((f(x), f(y)) \in \beta \Longrightarrow (x, y) \in \alpha).$$

The strongly extensional mapping f is called an *isomorphism* if it is injective and embedding, onto, isotone and reverse isotone. X and Y called *isomorphic*, in symbol $X \cong Y$, if exists an isomorphism between them. As in [6], a relation $\tau \subseteq X \times X$ is a *quasi-antiorder* on X if and only if

$$\tau\ \subseteq\ \alpha\,(\subseteq \neq),\ \tau\ \subseteq\ \tau * \tau,\ (\tau \cap \tau^{-1} = \emptyset).$$

2 Preliminaries

The first proposition gives some information about quasi-antiorder:

Lemma 2.1 ([7], Lemma 1; [6], Lemma 1) *Let $(X, =, \neq)$ be an anti-ordered and τ is a quasi-antiorder on X. Then, the relation $q = \tau \cup \tau^{-1}$ is a coequality relation on X, and $X/q = \{aq : a \in X\}$ with the anti-order θ, defined by $(aq, bq) \in \theta \Longleftrightarrow (a, b) \in \tau$ $(a, b \in X)$, is an anti-ordered set and $\pi : X \longrightarrow X/q$, defined by $\pi(a) = aq$, is an reverse isotone strongly extensional mapping from X onto X/q.*

Lemma 2.2 ([6], Lemma 2; [9], Theorem 5) *Let $(X, =, \neq, \alpha)$ and $(Y, =, \neq, \beta)$ be anti-ordered sets and $\varphi : X \longrightarrow Y$ an reverse isotone strongly extensional mapping. Then,*

$$\varphi^{-1}(\beta) = \{(a, b) \in X \times X : (\varphi(a), \varphi(b)) \in \beta\}$$

is a quasi-antiorder on X *with* $\varphi^{-1}(\varphi)\cup(\varphi^{-1}(\varphi)) = Coker\varphi$, *and* $X/Coker\varphi \cong Im\varphi$ *as anti-ordered sets.*

Lemma 2.3 ([9], Theorem 6) *Let* $(X, =, \neq, \alpha)$ *and* $(Y, =, \neq, \beta)$ *be anti-ordered sets and* $\varphi : X \longrightarrow Y$ *an reverse isotone strongly extensional mapping and* ρ *a quasi-antiorder on* X*. Then,* $\rho \supseteq \varphi^{-1}(\beta)$ *if and only if there is a unique reverse isotone strongly extensional mapping* ψ *from* $X/Coker\varphi$ *to* T *such that* $\varphi = \psi \circ \pi$*. Moreover* $Im\varphi = Im\psi$ *.*

Lemma 2.4 ([7], Theorem 1; [9], Theorem 8) *Let* $(X, =, \neq, \alpha)$ *be a set,* ρ *and* σ *quasi-antiorders on* X *such that* $\sigma \subseteq \rho$*. Then, the relation* σ/ρ*, defined by*

$$\sigma/\rho = \{(x(\rho \cup \rho^{-1}), y(\rho \cup \rho^{-1})) \in X/(\rho \cup \rho^{-1}) \times X/(\rho \cup \rho^{-1}) : (x, y) \in \sigma\},$$

is a quasi-antiorder on $X/(\rho \cup \rho^{-1})$ *and*

$$(X/(\rho \cup \rho^{-1})/((\sigma/\rho) \cup (\sigma/\rho)^{-1}) \cong X/(\sigma \cup \sigma^{-1})$$

holds as anti-ordered sets.

Lemma 2.5 ([6], Theorem 3; [7], Theorem 2) *Let* $(X, =, \neq)$ *be a set with apartness,* σ *a quasi-antiorder on* X*. Let* $\mathbf{A} = \{\tau : \tau$ *is quasi-antiorder on* X *such that* $\tau \subseteq \sigma\}$*. Let* $\mathbf{B}$ *be the set of all quasi-antiorders on* X/q*, where* $q = \sigma \cup \sigma^{-1}$*. For* $\tau \in \mathbf{A}$ *, we define a relation* $\tau" = \{(aq, bq) \in X/q \times X/q : (a, b) \in \tau\}$*. The mapping* $\psi : \mathbf{A} \longrightarrow \mathbf{B}$ *defined by* $\psi(\tau) = \tau"$ *is strongly extensional, injective and surjective mapping from* $\mathbf{A}$ *onto* $\mathbf{B}$ *and for* τ_1, τ_2 $\in \mathbf{A}$ *we have* $\tau_1 \subseteq \tau_2$ *if and only if* $\psi(\tau_1) \subseteq \psi(\tau_2)$*.*

3 Definitions and basic properties

Let $(X, =, \neq)$ be a set with apartness, q be a coequality relation on X and α be an anti-order relation on X. With it is associated the following relative $((X/q, =_1, \neq_1), \theta)$ where $\theta = \pi \circ \alpha \circ \pi^{-1}$. In [8] giving an answer on question "When the relation θ , defined above, is an anti-order relation on X/q?" we find necessary and sufficient conditions that the relation $\pi \circ \alpha \circ \pi^{-1}$ is an anti-order relation on X/q.

Lemma 3.1 ([8], Theorem 4) *Let* q *be a coequality relation in anti-ordered set* $(X, =, \neq, \alpha)$*. Then, the relation* $\theta = \pi \circ \alpha \circ \pi^{-1}$ *is an anti-order relation on factor-set* X/q *if and only if the relation* $\tau = Ker\pi \circ \alpha \circ Ker\pi$ *is a quasi-antiorder relation on* X *such that* $\tau \cup \tau^{-1} = q$*.*

By definition, for a quasi-antiorder ρ on an anti-ordered set $(X,=,\neq,\alpha)$ holds $\rho\subseteq\alpha$. Opposite inclusion does not hold, but result in Theorem 3.1 is a motive for introducing of the following new notion:

Definition 1 Let $(X,=,\neq,\alpha)$ be an anti-ordered set. A quasi-antiorder ρ on X is called a *quotient quasi-antiorder* (abbreviated to Q-quasi-antiorder) on X if holds

$$\alpha\subseteq Ker\pi\circ\rho\circ Ker\pi.$$

Let $\varphi:(X,=,\neq,\alpha)\longrightarrow(Y,=,\neq,\beta)$ be a strongly extensional reverse isotone mapping between anti-ordered sets. Then, by Lemma 2.2, the relation $\varphi^{-1}(\beta)$ is a quasi-antiorder on X with $\varphi^{-1}(\varphi)\cup(\varphi^{-1}(\beta))=Coker\varphi$, and $X/Coker\varphi\cong Im\varphi$ as anti-ordered sets. Besides, holds $\varphi^{-1}(\beta)\subseteq\alpha$ because φ is a reverse isotone mapping. A little generalization of notion introduced in the Definition 1 is the following notion:

Definition 2 Let $(X,=,\neq,\alpha)$ and $(Y,=,\neq,\beta)$ be anti-ordered sets. A reverse isotone strongly extensional mapping $\varphi:X\longrightarrow Y$ is called a *quotient anti-ordered mapping* (abbreviated to QA-mapping) of X to Y if holds

$$\alpha\subseteq Ker\pi\circ\varphi^{-1}(\beta)\circ Ker\varphi.$$

In the case when φ is onto, T is called a *quotient anti-ordered set* of S.

In the following theorem a characteristic of Q-quasi-antiorder is present:

Theorem 3.2 *Let $(X,=,\neq)$ be an anti-ordered set and ρ a Q-quasi-antiorder on X. Then $\pi:X\longrightarrow X/(\rho\cup\rho^{-1})$ is a QA-mapping from X onto $X/(\rho\cup\rho^{-1})$. Thus, $X/(\rho\cup\rho^{-1})$ is a quotient anti-ordered set of X.*

Proof: Let ρ is a Q-quasi-antiorder relation on X. Then $q=\rho\cup\rho^{-1}$ is a coequality relation on X and θ on X/q, defined by $(aq,bq)\in\theta\Longleftrightarrow(a,b)\in\rho$, is an anti-order on X/q and the mapping $\pi:X\longrightarrow X/q$, defined by $\pi(a)=aq$ $(a\in X)$, is a strongly extensional reverse isotone mapping from X onto X/q by Lemma 2.1. Since ρ is a Q - quasi-antiorder relation on X, then the inclusion $\alpha\subseteq Ker\pi\circ\rho\circ Ker\pi$ holds. Besides, since $\rho=\theta^{-1}(\theta)$, we have $\alpha\subseteq Ker\pi\circ\pi^{-1}(\theta)\circ Ker\pi$. Therefore, π is a QA - mapping from X onto X/q. $\square$

In the next assertion we give a connection between QA-mappings and Q-quasi-antiorders on anti-ordered sets.

Theorem 3.3 *Let $(X,=,\neq,\alpha)$ and $(Y,=,\neq,\beta)$ be anti-ordered sets and $\varphi:X\longrightarrow Y$ a strongly extensional reverse isotone QA-mapping. Then,*

$\varphi^{-1}(\beta)$ is a Q-quasi-antiorder on X with $\varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta))^{-1} = Coker\varphi$.
Proof: Let $\varphi : X \longrightarrow Y$ be a strongly extensional reverse isotone QA-mapping. Then, by Lemma 2.2, $\varphi^{-1}(\beta)$ is a quasi-antiorder on X such that $\varphi^{-1}(\beta) \cup (\varphi^{-1}(\beta)) = Coker\varphi$. Since, by Definition 2, we have $\alpha \subseteq Ker\varphi \circ \varphi^{-1}(\beta) \circ Ker\varphi$, then $\varphi^{-1}(\beta)$ is Q - quasi-antiorder relation on X. □

4 Isomorphism theorems

In this section we present two isomorphism theorems on QA-mappings and Q-quasi-antiorder.

Theorem 4.1 (First Isomorphism theorem) *Let $(X, =, \neq, \alpha)$ and $(Y, =, \neq, \beta)$ be anti-ordered sets and $\varphi : X \longrightarrow Y$ a QA-mapping and ρ a Q-quasi-antiorder on X. Then, $\rho \supseteq \varphi^{-1}(\beta)$ if and only if there is a unique QA-mapping $\psi : X/(\rho \cup \rho^{-1}) \longrightarrow Y$ such that $\varphi = \psi \circ \pi$. Moreover, $Im\varphi = Im\psi$.*
Proof:
$(\Longrightarrow)$ Let $(\alpha \supseteq)\rho \supseteq \varphi^{-1}(\beta)$. By Lemma 2.3, there exists a unique reverse isotone strongly extensional mapping ψ from $X/Coker\varphi$ to T such that $\varphi = \psi \circ \pi$ with $Im\varphi = Im\psi$. Further on, $q = \rho \cup \rho^{-1}$ is a coequality on X and $\pi : X \longrightarrow X/q$ is a strongly extensional reverse isotone QA-mapping from X onto $(X/q, =_1, \neq_1, \theta)$ by Theorem 3.2. Besides, let $(aq, bq) \in \theta$ be an arbitrary element. Then, by definition of θ, we have $(a, b) \in \rho$. Since ρ is a Q-quasi-antiorder on X and φ is QA-mapping, the inclusion

$$\rho \subseteq \alpha \subseteq Ker\varphi \circ \varphi^{-1}(\beta) \circ Ker\varphi$$

is valid. Thus, there exist elements $x, y \in X$ such that $(a, x) \in Ker\varphi$ and $(y, b) \in Ker\varphi$, i.e. there exist elements $x, y \in X$ such that

$$\varphi(a) = \varphi(x) \wedge (x, y) \in \varphi^{-1}(\beta) \wedge \varphi(y) = \varphi(b)$$

i.e. we have element $x, y \in X$ such that

$$\psi(\pi(a)) = \psi(\pi(x)) \wedge (\pi(x), \pi(y)) \in \psi^{-1}(\beta) \wedge \psi(\pi(y)) = \psi(\pi(b)).$$

Finally, we have

$$(\pi(a), \pi(b)) \in Ker\psi \circ \psi^{-1}(\beta) \circ Ker\psi.$$

So, the inclusion

$$\theta \subseteq Ker\psi \circ \psi^{-1}(\beta) \circ Ker\psi$$

is proved. Therefore, $\psi : (X/q, =_1, \neq_1, \theta) \longrightarrow (Y, =, \neq, \beta)$ is QA-mapping.
$(\Longleftarrow)$ This part of proof immediately follows from Lemma 2.3. □

Theorem 4.2 (Second Isomorphism Theorem) *Let* $(X, =, \neq, \alpha)$ *be a set,* ρ *and* σ *Q-quasi-antiorders on* X *such that* $\sigma \subseteq \rho$. *Then the relation* σ/ρ *, defined by*

$$\sigma/\rho = \{(x(\rho \cup \rho^{-1}), y(\rho \cup \rho^{-1})) \in X/(\rho \cup \rho^{-1}) \times X/(\rho \cup \rho^{-1}) : (x, y) \in \sigma\},$$

is a Q-quasi-antiorder on $X/(\rho \cup \rho^{-1})$ *and*

$$(X/(\rho \cup \rho^{-1})/((\sigma/\rho\cup)(\sigma/\rho)^{-1}) \cong X/(\sigma \cup \sigma^{-1})$$

holds as anti-ordered sets.

Proof: By Lemma 2.4, the relation σ/ρ is a quasi-antiorder on $(X/(\rho\cup\rho^{-1}), =_1, \neq_1, \theta)$ and

$$(X/(\rho \cup \rho^{-1})/((\sigma/\rho) \cup (\sigma/\rho)^{-1}) \cong X/(\sigma \cup \sigma^{-1})$$

holds as anti-ordered sets. Let π_t be the natural strongly extensional reverse isotone mapping from $(X/(\rho \cup \rho^{-1})$ onto $(X/(\rho \cup \rho^{-1})/((\sigma/\rho) \cup (\sigma/\rho)^{-1})$. We need to prove only that σ/ρ is Q-quasi-antiorder on $X/(\rho \cup \rho^{-1})$, i.e. we need to prove

$$\theta \subseteq Ker\pi_t \circ \sigma/\rho \circ Ker\pi_t.$$

Let $q = \rho \cup \rho^{-1}$, $p = \sigma \cup \sigma^{-1}$, $t = (\sigma/\rho) \cup (\sigma/\rho)^{-1}$ and (aq, bq) be an arbitrary element of θ. Then, by definition of θ, $(a, b) \in \rho \subseteq \alpha$. Since σ is a Q-quasi-antiorder on X, we have

$$(a, b) \in \rho \subseteq \alpha \subseteq Ker\pi_\sigma \circ \sigma \circ Ker\pi_\sigma.$$

Thus, there exist elements x, y of X such that

$$\pi_\sigma(a) =_2 \pi_\sigma(x) \wedge (x, y) \in \sigma \wedge \pi_\sigma(y) =_2 \pi_\sigma(b),$$

i.e. such that

$$(a, x) \bowtie p \supseteq \sigma \wedge (xq, yq) \in \sigma/\rho \wedge (y, b) \bowtie psupseteq\, \sigma.$$

Further on, let (uq, vq) be an arbitrary element of t, i.e. let (u, v) be an arbitrary element of σ . Thus, we have $((u, a) \in b\sigma \vee (a, x) \in \sigma \vee (x, v) \in \sigma)$. So, we conclude

$$((uq)t \neq_3 (xq)t) \wedge ((xq)t \neq_3 (vq)t)$$

because the case $(a, x) \in \sigma \subseteq p$ is impossible. Therefore, we have $(aq, xq) \bowtie t$. Analogously, we have that $(yq, bq) \in t$ also. Finally, we have

$$(aq)t =_3 (xq)t \wedge (xq, yq) \in \sigma/\rho \wedge (yq)t =_3 (bq)t,$$

i.e. we have $(aq, bq) \in Ker\pi_t \circ \sigma/\rho \circ Ker\pi_t$. □

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